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USING ELZAKI TRANSFORM TO SOLVING THE KLEIN-GORDON EQUATION

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Abstract. In this paper, Elzaki transform and variational iteration method were applied to solve non-homogeneous Klein-Gordon equation. The reason of using Elzaki transform is that one obtains the exact solution of this problem. By the variational iteration method an iteration sequence is achieved which will converge to analytical solution of the Klein-Gordon equation.

Keywords: Elzaki transform, Klein-Gordon equation, variational iteration.

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1. Introduction

A large number of physical and engineering problems lead to the partial differential equations. Such equations are often identified as differential equations with initial and boundary conditions. Usually for such equations, finding an analytical solution is either difficult or even sometimes impossible [7].

The Elzaki transform method which is based on Fourier transform, introduced by tarig Elzaki in 2010 (see [5]). In 2011 this method has been used to solve Integro-differential equation and partial differential equations [5] and [6]. The Elzaki transform method is a useful and simple method for solving partial differential equations. One of the advantages of this method is that we can exactly obtain the solution of the equations. In this paper, we use Elzaki transform method to solve non-homogeneous Klein-Gordon equation since the solution is obtained exactly and simply. One can see the efficiency of this method by solving the Klein-Gordon equations.

The variational iteration method is capable for solving different kinds of engineering and scientific problems that was first introduced by Inokuti in 1978 and then, generalized by He in 1999 and Hesameddini in 2009(see [1, 2, 3]).

Using this method, we can achieve a sequence with high convergence speed in which, at last, an exact solution of the problem will be achieved. This method has been invented for solving non-linear or even strictly non-linear problems. The Klein-Gordon equation is considered as follows:

$$u_{tt} - u_{xx} + b_1 u = f(x,t),$$

with initial conditions:

$$\begin{cases} u(x,0) = a_0(x), \\ u_t(x,0) = a_1(x), \end{cases} \tag{1}$$

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where b_1 is a real number and f is a known non-linear function. In this paper, we will solve non-homogeneous Klein-Gordon equation through Elzaki transform and variational iteration method, also comparing of these method will be demonstrated.

2. Elzaki transform

In this section we introduce Elzaki transform and state some useful formula of its properties. The Elzaki transform of the continues function $f(t)$ is defined as (see[6]):

$$E[f(t)] = T(\nu) = \nu \int_0^\infty f(t) \exp^{\frac{-t}{\nu}} dt, \qquad t > 0, \quad \nu \in (-k_1, k_2), \tag{2}$$

and for the two variable function $f(x,t)$, we have:

$$E[f(x,t)] = T(\nu) = \nu \int_0^\infty f(x,t) \exp^{\frac{-t}{\nu}} dt, \qquad t > 0, \quad \nu \in (-k_1, k_2),$$

also the Elzaki transform of partial derivatives of $f(x,t)$ are defined as follows (see[6]):

$$\begin{aligned} &E[\tfrac{\partial f}{\partial t}(x,t)] = \tfrac{T(x,\nu)}{\nu} - \nu f(x,0), \\ &E[\tfrac{\partial f}{\partial x}(x,t)] = \tfrac{d}{dx}[T(x,\nu)], \\ &E[\tfrac{\partial^2 f}{\partial x^2}(x,t)] = \tfrac{d^2}{dx^2}[T(x,\nu)], \\ &E[\tfrac{\partial^2 f}{\partial t^2}(x,t)] = \tfrac{T(x,\nu)}{\nu^2} - f(x,0) - \nu \tfrac{\partial f}{\partial t}(x,0). \end{aligned} \tag{3}$$

We also remined a useful properties of this transformation which will be used later. Let $f(t)$ and $g(t)$ having Elzaki transforms $M(\nu)$ and $N(\nu)$, then the Elzaki transform of the convolution of f and g is given by (for more detail see[5]):

$$E[(f * g)(t)] = \frac{1}{\nu} M(\nu) N(\nu). \tag{4}$$

Example 2.1.
Consider the Klein-Gordon equation:

$$u_{tt} - u_{xx} - 2u = cos(x), \tag{5}$$

with initial conditions:

$$u(x,0) = 1 + x, \qquad\qquad u_t(x,0) = 0.$$

Let $T(x,\nu)$ be the Elzaki transform of $u(x,t)$. Then, the Elzaki transform of the equation (5) is:

$$\frac{1}{\nu^2} T(x,\nu) - u(x,0) - \nu u_t(x,0) - 2T(x,\nu) + \nu^2 cosx = \frac{d^2}{dx^2} T(x,\nu),$$

or equivalently,

$$\frac{d^2}{dx^2} T(x,\nu) + \frac{2\nu^2 - 1}{\nu^2} T(x,\nu) = \nu^2 cosx - (1 + x).$$

By using the inverse operator, result in:

$$[\frac{\nu^2 D^2 + 2\nu^2 - 1}{\nu^2}] T(x,\nu) = \nu^2 cosx - (1 + x),$$

where $D^2 \equiv \frac{d^2}{dx^2}$; Then, by solving this problem, one obtains:

$$T(x,\nu) = \tfrac{\nu^2}{\nu^2 D^2 + 2\nu^2 - 1}(\nu^2 cosx - (1 + x)) = \tfrac{\nu^2}{\nu^2 - 1} \nu^2 cosx - \tfrac{\nu^2}{\nu^2 - 1}(1 + x).$$

To obtain the inverse Elzaki transformation, we use the convolution theorem [6] for

$$\frac{\nu^2}{\nu^2-1}\nu^2 cosx,$$

then we get:

$$E^{-1}[\frac{\nu^2}{\nu^2-1}\nu^2 cosx] = \int_0^t (-sinh(t-x))dx = (1-cosh(t)), \tag{6}$$

and

$$E^{-1}[\frac{\nu^2}{\nu^2-1}(-(1+x))] = (1+x)cosh(t). \tag{7}$$

Therefore, by (6) and (7) the solution is $u(x,t) = cosh(t)(1+x) + (1-\cosh(t))\cos(x)$. In figure 1 the graph of this function for $-5 < t < 5$ and $-5 < x < 5$ will be shown.

Example 2.2.

Consider

$$u_{tt} = u_{xx} - 2sinxsint + 2u, \tag{8}$$

subject to the initial conditions:

$$u(x,0) = 0, \qquad u_t(x,0) = sinx.$$

Let $T(x,\nu)$ be the Elzaki transform of $u(x,t)$. Then, the Elzaki transform of equation (8) is:

$$\frac{1}{\nu^2}T(x,\nu) - u(x,0) - \nu u_t(x,0) = \frac{d^2}{dx^2}T(x,\nu) - 2sinx\frac{\nu^3}{1+\nu^2} + 2T(x,\nu),$$

or equivalently

$$\frac{d^2}{dx^2}T(x,\nu) + \frac{2\nu^2-1}{\nu^2}T(x,\nu) = sinx(\frac{2\nu^3}{1+\nu^2} - \nu).$$

By using inverse operator, it results in:

$$(D^2 + \frac{2\nu^2-1}{\nu^2})T(x,\nu) = sinx(\frac{2\nu^3}{1+\nu^2} - \nu).$$

Thus we obtain the following relation:

$$T(x,\nu) = sinx(\frac{2\nu^3}{1+\nu^2} - \nu)(\frac{\nu^2}{\nu^2 D^2 + 2\nu^2 - 1}) = sinx(\frac{2\nu^3}{1+\nu^2} - \nu)(\frac{\nu^2}{\nu^2-1}),$$

To obtain the inverse Elzaki transformation, we must use the convolution theorem; then one gets:

$$E^{-1}[(\tfrac{2\nu^3}{1+\nu^2} - \nu)(\tfrac{\nu^2}{\nu^2-1})]$$

$$= -2\int_0^t sinxsinh(t-x)dx - \int_0^t cosh(t-x)dt$$

$$= \tfrac{-1}{2}cos(x)(\exp(x-t) - \exp(t-x)) + \tfrac{1}{2}sin(x)(\exp(x-t) + \exp(t-x))|_0^t$$

$$= sint - sinh(t) + sinh(t) = sint.$$

Therefore, $u(x,t) = sinxsint$, which is the exact solution of equation (8). In figure 2 the graph of this function for $-3 < t < 3$ and $-3 < x < 3$ will be shown.

3. Variational iteration method

Consider the nonlinear equation

$$L(u) + N(u) = g.$$

The variational iteration method is in this way (see [4]):

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda[L[u_n(x)] + N[\widetilde{u_n(x)}] - g(x)]dx, \tag{9}$$

where, L is a linear operation, N is a non-linear term, $g(x)$ is a given known function and λ is an essential lagrange multiplier which is calculated by using variational theory [2]. In equation (9), n represents the nth approximation of the real solution u and $\widetilde{u_n(x)}$ is the part in which variations are limited and this means that variations are equal to zero in $\widetilde{u_n}$ ([3]).

$$\begin{cases} u_{n+1}(t) = u_n(t) + \int_0^t \lambda[L[u_n(x)] + N[\widetilde{u_n(x)}] - g(x)]dx, \\ \delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda[L[u_n(x)] - g(x)]dx. \end{cases} \tag{10}$$

Example 3.1. Consider the following equation

$$u_{tt} - u_{xx} - 2u = cos(x),$$

with the initial conditions:

$$u(x,0) = 1 + x, \qquad u_t(x,0) = 0.$$

We present the function $u(x,t)$ by the variational iteration method through the following equation([1]):

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s)[u_{n(ss)}(x,s) - \widetilde{u_{n(xx)}(x,s)} - 2\widetilde{u_n(x,s)} + cos(x)]ds.$$

In order to obtain $\lambda(s)$, we consider the following variational relationship by supposing $\delta u_n(0) = 0$ and $\delta\widetilde{u_n} = 0$, as:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(s)[u_{n(ss)}(x,s)]ds. \tag{11}$$

Using integration by part of (11) and stability condition, one obtains:

$$\begin{aligned} \delta u_{n+1}(x,t) &= \delta u_n(x,t) + \lambda(s)\delta u_{n(s)}|_{s=t} - \delta \int_0^t \dot{\lambda}(s) u_{n(s)} ds \\ &= \delta u_n(x,t) + \lambda(s)\delta u_n(s)|_{s=t} - \dot{\lambda}(s)\delta u_{n(s)}|_{s=t} + \delta \int_0^t \ddot{\lambda}(s) u_n ds, \end{aligned} \tag{12}$$

such that:

$$\begin{cases} 1 - \dot{\lambda}(s)|_{s=t} = 0, \\ \lambda(s)|_{s=t} = 0. \end{cases} \tag{13}$$

Therefore, $\lambda(s) = (s - t)$.
Therefore, we conclude the following iteration relationship:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)[u_{n(ss)}(x,s) - u_{n(xx)}(x,s) - 2u_n(x,s) + \cos(x)]ds. \tag{14}$$

Inserting the initial condition, it results:

$$u(x,0) = 1 + x, \qquad u_t(x,0) = 0.$$

The first guess for u is: $u_0(x,t) = 1 + x$. Then, by (13) one obtains:

$$\begin{cases} u_1 = (1+x) + \int_0^t (s-t)[-2(1+x) + \cos(x)]ds = (1+x) + t^2(1+x) - \frac{t^2}{2}\cos(x), \\ u_2 = (1+x) + t^2(1+x) + \frac{t^4}{6} - \frac{t^2}{2}\cos(x) - \frac{t^4}{24}\cos(x), \\ u_3 = u_2 + \frac{t^6}{90}(1+x) - \frac{t^6}{6!}\cos(x). \end{cases} \tag{15}$$

Thus where $n \to \infty$, one can conclude that:

$$u_n \to \cosh(t)(1+x) + (1 - \cosh(t))\cos(x).$$

In section (2), we solved non-homogeneous Klein-Gordon equation with Elzaki transformation and we obtain its exact solution very simplicity. Also in this section, we have solved the Klein-Gordon equation by using variational iteration method.

Example 3.2.

Consider the following equation:

$$u_{tt} = u_{xx} - 2sinxsint + 2u,$$

with the initial conditions:

$$u(x,0) = 0, \qquad u_t(x,0) = sinx.$$

We obtain its solution $u(x,t)$ by the variational iteration method from the following sequence:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s)[u_{n(ss)}(x,s) - \widetilde{u_{n(xx)}(x},s) - 2\widetilde{u_n(x,s)} + 2\widetilde{sinxsins}]ds. \tag{16}$$

The same as example (3.1), taking $\lambda(s) = (s - t)$, we can construct the following iteration relationship:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)[u_{n(ss)}(x,s) - u_{n(xx)}(x,s) - 2u(x,s) + 2sinxsins]ds. \tag{17}$$

Considering the initial conditions:

$$u(x,0) = 0, \qquad u_t(x,0) = sinx,$$

our first guess will be as follow:

$$u_0(x,t) = t\sin(x).$$

Therefore from (16) we obtain:

$$\begin{cases} u_1 = -tsinx + \frac{t^3}{6}sinx + 2sintsinx, \\ u_2 = tsinx - \frac{t^3}{6}sinx + \frac{t^5}{5!}sinx, \\ u_3 = -tsinx + \frac{t^3}{6}sinx - \frac{t^5}{5!}sinx + \frac{t^7}{7!}sinx + 2sintsinx. \end{cases} \tag{18}$$

As it is observed, by repeating the above processes, it results:

$$u_n(x,t) = (-t + \frac{t^3}{3!} - \frac{t^5}{5!} + \ldots + \frac{t^{2n+1}}{(2n+1)!})\sin(x) + 2\sin(t)\sin(x),$$

where n is odd and

$$u_n(x,t) = (-t + \frac{t^3}{3!} - \frac{t^5}{5!} + \ldots + \frac{t^{2n+1}}{(2n+1)!})\sin(x),$$

if n is even. Therefore, if $n \to \infty$, one obtains:

$$u_n(x,t) = -\sin(t)\sin(x) + 2\sin(t)\sin(x) = \sin(t)\sin(x),$$

where n is an odd number and

$$u_n(x,t) = \sin(t)\sin(x),$$

where n is an even number which converges to the exact solution.

4. Conclusions

In solving Klein-Gordon problem using variational iteration method, Lagrang multiplier can be calculated easily and the obtained sequence is an recursive sequence which is converged to the real solution of this problem. This illustrates the effectiveness of this method in this specific problem. By using Elzaki transformation, we can obtain the exact solution of this equation, which is one of the advantages of this method. Comparing Elzaki transform method and variational iteration method, we can see that for linear and non-linear differential equation Elzaki method is a very useful tool.

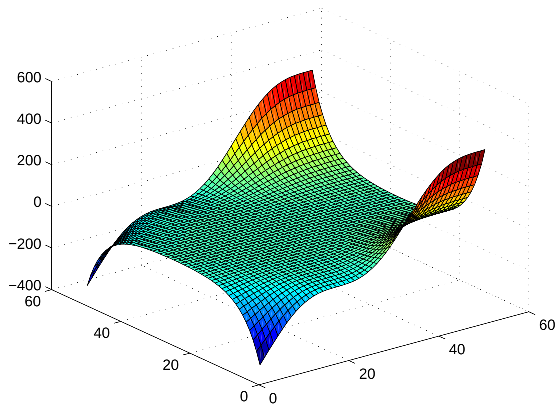

Figure 1. u(x,t)=cosh(t)(1+x)+(1-cosh(t))cos(x)

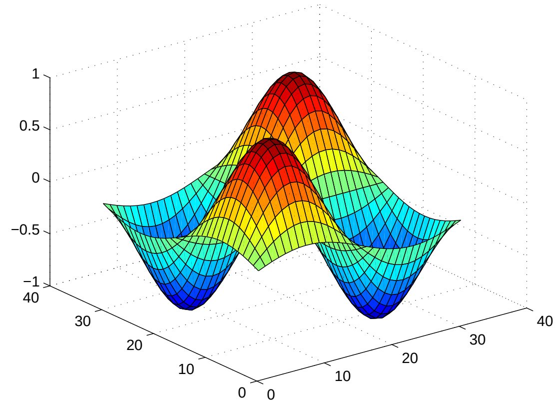

Figure 2. u(x,t)=sin(t)sin(x)

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